The Newtonian limit of fourth and higher order gravity*

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Abstract

We consider the Newtonian limit of the theory based on the Lagrangian

$$\mathcal{L} = \left(R + \sum_{k=0}^{p} a_k \, R \Box^k R \right) \sqrt{-g} \, .$$

The gravitational potential of a point mass turns out to be a combination of Newtonian and Yukawa terms. For sixth-order gravity ($p = 1$) the coefficients are calculated explicitly. For general p one gets

$$\Phi = m/r \left(1 + \sum_{i=0}^{p} c_i \exp(-r/l_i) \right)$$

with $\sum_{i=0}^{p} c_i = 1/3$. Therefore, the potential is always unbounded near the origin.

Wir betrachten den Newtonschen Grenzwert der durch den Lagrangian

$$\mathcal{L} = \left(R + \sum_{k=0}^{p} a_k \, R \Box^k R \right) \sqrt{-g}$$

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beschriebenen Theorie. Das Gravitationspotential einer Punktmasse hat die Gestalt von Kombinationen von Newton- und Yukawa-Termen. Für Gravitationsgleichungen sechster Ordnung ($p = 1$) wird das Fernfeld von Punktmassen explizit angegeben. Für allgemeine p wird die Summe der Koeffizienten vor den Yukawa-Termen durch $\sum_{i=0}^{p} c_i = 1/3$ beschrieben. Also ist das Potential stets unbeschränkt.

Key words: gravitational theory - high-order gravity - Newtonian limit
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1 Introduction

One-loop quantum corrections to the Einstein equation can be described by curvature-squared terms and lead to fourth-order gravitational field equations; their Newtonian limit is described by a potential "Newton + one Yukawa term", cf. e.g. STELLE (1978) and TEYSSANDIER (1989). A Yukawa potential has the form $\exp(-r/l)/r$ and was originally used by YUKAWA (1935) to describe the meson field.

Higher-loop quantum corrections to the Einstein equation are expected to contain terms of the type $R\Box^k R$ in the Lagrangian, which leads to a gravitational field equation of order $2k+4$, cf. GOTTLÖBER et al. (1990). Some preliminary results to this type of equations are already due to BUCHDAHL (1951).

In the present paper we deduce the Newtonian limit following from this higher order field equation. The Newtonian limit of General Relativity Theory is the usual Newtonian theory, cf. e.g. DAUTCOURT (1964) or STEPHANI (1977). From the general structure of the linearized higher-order field equation, cf. SCHMIDT (1990), one can expect that for this higher-order gravity the far field of the point mass in the Newtonian limit is

the Newtonian potential plus a sum of different Yukawa terms. And just this form is that one discussed in connection with the fifth force, cf. GERBAL and SIROUSSE-ZIA (1989), STACEY et al. (1981) and SANDERS (1984, 1986).

Here we are interested in the details of this connection between higher-order gravity and the lengths and coefficients in the corresponding Yukawa terms.

2 Lagrangian and field equation

Let us start with the Lagrangian

$$\mathcal{L} = \left(R + \sum_{k=0}^{p} a_k \, R \Box^k R \right) \cdot \sqrt{-g} \, , \qquad a_p \neq 0 \, . \tag{1}$$

In our considerations we will assume that for the gravitational constant G and for the speed of light c it holds $G = c = 1$. This only means a special choice of units. In eq. (1), R denotes the curvature scalar, $\Box$ the D'Alembertian, and g the determinant of the metric. Consequently, the coefficient a_k has the dimension "length to the power $2k+2$".

The starting point for the deduction of the field equation is the principle of minimal action. A necessary condition for it is the stationarity of the action:

$$-\frac{\delta \mathcal{L}}{\delta g_{ij}} = 8\pi T^{ij} \sqrt{-g} \, ,$$

where T^{ij} denotes the energy-momentum tensor. The explicit equations for

$$P^{ij} \sqrt{-g} = -\frac{\delta \mathcal{L}}{\delta g_{ij}}$$

are given by SCHMIDT (1990). Here we only need the linearized field equation. It reads, cf. GOTTLÖBER et al. (1990)

$$P^{ij} \equiv R^{ij} - \frac{R}{2} g^{ij} + 2 \sum_{k=0}^{p} a_k [g^{ij} \Box^{k+1} R - \Box^k R^{;ij}] = 8\pi T^{ij} \, , \tag{2}$$

and for the trace it holds:

$$g_{ij} \cdot P^{ij} = -\frac{n-1}{2} R + 2n \sum_{k=0}^{p} a_k [g^{ij} \Box^{k+1} R] = 8\pi T \,. \tag{3}$$

n is the number of spatial dimensions; the most important application is of cause $n = 3$. From now on we put $n = 3$.

3 The Newtonian limit

The Newtonian limit is the weak-field static approximation. So we use the linearized field equation and insert a static metric and an energy-momentum tensor

$$T_{ij} = \delta_i^0 \ \delta_j^0 \, \rho \,, \qquad \rho \geq 0 \tag{4}$$

into eq. (2).

Without proof we mention that the metric can be brought into spatially conformally flat form and so we use

$$g_{ij} = \eta_{ij} + f_{ij} \,,$$
$$\eta_{ij} = \mathrm{diag}(1, \, -1, \, -1, \, -1) \qquad \text{and}$$
$$f_{ij} = \mathrm{diag}(-2\Phi, \, -2\Psi, \, -2\Psi, \, -2\Psi) \,.$$

So the metric equals

$$ds^2 = (1 - 2\Phi) dt^2 - (1 + 2\Psi)(dx^2 + dy^2 + dz^2) \,, \tag{5}$$

where Φ and Ψ depend on x, y and z.

Linearization means that the metric g_{ij} has only a small difference to η_{ij}; quadratic expressions in f_{ij} and its partial derivatives are neglected.

We especially consider the case of a point mass. In this case it holds: $\Phi = \Phi(r)$, $\Psi = \Psi(r)$, with

$$r = \sqrt{x^2 + y^2 + z^2} \,,$$

because of spherical symmetry and $\rho = m\delta$. Using these properties, we deduce the field equation and discuss the existence of solutions of the above mentioned type.

At first we make some helpful general considerations: The functions Φ and Ψ are determined by eq. (2) for $i = j = 0$ and the trace of eq. (2). If these two equations hold, then all other component-equations are automatically satisfied. For the 00-equation we need R_{00}:

$$R_{00} = -\Delta\Phi \,. \tag{6}$$

Here is as usual

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \,.$$

For the inverse metric we get

$$g^{ij} = \mathrm{diag}\,(1/(1-2\Phi),\ -1/(1+2\Psi),\ -1/(1+2\Psi),\ -1/(1+2\Psi))$$

and $1/(1-2\Phi) = 1 + 2\Phi + h(\Phi)$, where $h(\Phi)$ is quadratic in Φ and vanishes after linearization. So we get

$$g^{ij} = \eta^{ij} - f^{ij} \,.$$

In our coordinate system, f^{ij} equals f_{ij} for all i, j. For the curvature scalar we get

$$R = 2(2\Delta\Psi - \Delta\Phi) \,. \tag{7}$$

Moreover, we need expressions of the type $\Box^k R$. $\Box R$ is defined by $\Box R = R_{;ij}\, g^{ij}$, where ";" denotes the covariant derivative. Remarks: Because of linearization we may replace the covariant derivative with the partial one. So we get

$$\Box^k R = (\ 1)^k\, 2(-\Delta^{k+1}\Phi + 2\Delta^{k+1}\Psi) \tag{8}$$

and after some calculus

$$-\, 8\pi\rho = \Delta\Phi + \Delta\Psi \,. \tag{9}$$

We use eq. (9) to eliminate Ψ from the system. So we get by help of eq. (8) an equation relating Φ and $\rho = m\delta$.

$$-4\pi\left(\rho + 8\sum_{k=0}^{p} a_k(-1)^k\Delta^{k+1}\rho\right) = \Delta\Phi + 6\sum_{k=0}^{p} a_k(-1)^k\Delta^{k+2}\Phi\,. \tag{10}$$

In spherical coordinates it holds

$$\Delta\Phi = \frac{2}{r}\Phi_{,r} + \Phi_{,rr}\,,$$

because Φ depends on the radial coordinate r only.

We apply the following lemma: In the sense of distributions it holds

$$\Delta\left(\frac{1}{r}e^{-r/l}\right) = \frac{1}{rl^2}e^{-r/l} - 4\pi\delta\,. \tag{11}$$

Now we are ready to solve the whole problem. We assume

$$\Phi = \frac{m}{r}\left(1 + \sum_{i=0}^{q} c_i\exp(-r/l_i)\right)\,, \quad l_i > 0\,.$$

Without loss of generality we may assume $l_i \neq l_j$ for $i \neq j$. Then eq. (10) together with the lemma eq. (11) yield

$$\begin{aligned}8\pi\sum_{k=0}^{p} a_k(-1)^k\Delta^{k+1}\delta = &\sum_{i=0}^{q}\left(\frac{c_i}{t_i} + 6\sum_{k=0}^{p} a_k(-1)^k\frac{c_i}{t_i^{k+2}}\right)\frac{1}{rl^2}e^{-r/l_i}\\ &-4\pi\sum_{i=0}^{q}\left(c_i + 6\sum_{k=0}^{p} a_k(-1)^k\frac{c_i}{t_i^{k+1}}\right)\delta\\ &+24\pi\sum_{k=0}^{p}\sum_{j=k}^{p}\sum_{i=0}^{p} c_i a_j(-1)^{j+1}\frac{1}{t_i^{j-k}}\Delta^{k+1}\delta\end{aligned}$$

where $t_i = l_i^2$; therefore also $t_i \neq t_j$ for $i \neq j$. This equation is equivalent to the system (12, 13, 14)

$$\sum_{i=0}^{q} c_i = 1/3 \tag{12}$$

$$\sum_{i=0}^{q}\frac{c_i}{t_i^s} = 0 \qquad s = 1,\dots p \tag{13}$$

$$t_i^{p+1} + 6\sum_{k=0}^{p} a_k(-1)^k t_i^{p-k} = 0 \qquad i = 0,\dots q\,. \tag{14}$$

From eq. (14) we see that the values t_i represent $q+1$ different solutions of one polynomial. This polynominal has the degree $p+1$. Therefore $q \leq p$.

Now we use eqs. (12) and (13). They can be written in matrix form as

$$\begin{pmatrix} 1 \ldots 1 \\ 1/t_0 \ldots 1/t_q \\ \ldots \\ 1/t_0^p \ldots 1/t_q^p \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \ldots \\ c_q \end{pmatrix} = \begin{pmatrix} 1/3 \\ 0 \\ \ldots \\ 0 \end{pmatrix}$$

Here, the first $q+1$ rows form a regular matrix (Vandermonde matrix). Therefore, we get

$$1/t_i^j = \sum_{k=0}^{q} \lambda_{jk} \, / \, t_i^k \qquad j = q+1, \ldots p$$

with certain coefficients λ_{jk} i.e., the remaining rows depend on the first $q+1$ ones. If $\lambda_{j0} \neq 0$ then the system has no solution. So $\lambda_{j0} = 0$ for all $q+1 \leq j \leq p$. But for $q < p$ we would get

$$1/t_i^q = \sum_{k=1}^{q} \lambda_{q+1\,k} \, / \, t_i^{k-1}$$

and this is a contradiction to the above stated regularity. Therefore p equals q. The polynomial in (14) may be written as

$$6 \cdot \begin{pmatrix} 1 \; 1/t_0 \ldots 1/t_0^p \\ \ldots \\ 1 \; 1/t_p \ldots 1/t_p^p \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ \ldots \\ (-1)^p a_p \end{pmatrix} = \begin{pmatrix} -t_0 \\ \ldots \\ -\tau_p \end{pmatrix}$$

This matrix is again a Vandermonde one, i.e., there exists always a unique solution $(a_0, \ldots a_p)$ (which are the coefficients of the quantum corrections to the Einstein equation) such that the Newtonian limit of the corresponding gravitational field equation is a sum of Newtonian and Yukawa potential with prescribed lengths l_i. A more explicit form of the solution is given in the appendix.

4 Discussion

Let us give some special examples of the deduced formulas of the Newtonian limit of the theory described by the Lagrangian (1). If all the a_i vanish we get of course the usual Newton theory

$$\Phi = \frac{m}{r}, \quad \Delta\Phi = -4\pi\delta\,.$$

Φ and Ψ refer to the metric according to eq. (5). For $p = 0$ we get for $a_0 < 0$

$$\Phi = \frac{m}{r}\left[1 + \frac{1}{3}\,e^{-r/\sqrt{-6a_0}}\right]$$

(cf. STELLE 1978) and

$$\Psi = \frac{m}{r}\left[1 - \frac{1}{3}\,e^{-r/\sqrt{-6a_0}}\right]$$

(cf. SCHMIDT 1986). For $a_0 > 0$ no Newtonian limit exists.

For $p = 1$, i.e., the theory following from sixth-order gravity

$$\mathcal{L} = \left(R + a_0 R^2 + a_1 R\Box R\right)\sqrt{-g}\,,$$

we get

$$\Phi = \frac{m}{r}\left[1 + c_0 e^{-r/l_0} + c_1 e^{-r/l_1}\right]$$

and

$$\Psi = \frac{m}{r}\left[1 - c_0 e^{-r/l_0} - c_1 e^{-r/l_1}\right]$$

where

$$c_{0,1} = \frac{1}{6} \mp \frac{a_0}{2\sqrt{9a_0^2 + 6a_1}}$$

and

$$l_{0,1} = \sqrt{-3a_0 \pm \sqrt{9a_0^2 + 6a_1}}\,.$$

(This result is similar in structure but has different coefficients as in fourth-order gravity with included square of the Weyl tensor in the Lagrangian.)

The Newtonian limit for the degenerated case $l_0 = l_1$ can be obtained by a limiting procedure as follows: As we already know $a_0 < 0$, we can choose the length unit such that $a_0 = -1/3$. The limiting case $9a_0^2 + 6a_1 \to 0$ may be expressed by $a_1 = -1/6 + \epsilon^2$. After linearization in ϵ we get:

$$l_i = 1 \pm \sqrt{3/2}\,\epsilon\, c_i = 1/6 \pm 1/(6\sqrt{6}\epsilon)$$

and applying the limit $\epsilon \to 0$ to the corresponding fields Φ and Ψ we get

$$\Phi = m/r\{1 + (1/3 + r/6)e^{-r}\}$$
$$\Psi = m/r\{1 - (1/3 + r/6)e^{-r}\}\,.$$

For the general case $p > 1$, the potential is a complicated expression, but some properties are explicitly known, (these hold also for $p = 0, 1$). One gets

$$\Phi = m/r\left(1 + \sum_{i=0}^{p} c_i \exp(-r/l_i)\right)$$

and

$$\Psi = m/r\left(1 - \sum_{i=0}^{p} c_i \exp(-r/l_i)\right)$$

where $\sum c_i = 1/3$; $\sum$ means $\sum_{i=0}^{p}$ and l_i and c_i are (up to permutation of indices) uniquely determined by the Lagrangian.

There exist some inequalities between the coefficients a_i, which must be fulfilled in order to get a physically acceptable Newtonian limit. By this phrase we mean that besides the above conditions, additionally the fields Φ and Ψ vanish for $r \to \infty$ and that the derivatives $d\Phi/dr$ and $d\Psi/dr$ behave like $O(1/r^2)$. These inequalities express essentially the fact that the l_i are real, positive, and different from each other. The last of these three conditions can be weakened by allowing the c_i to be polynomials in r instead of being constants, cf. the example with $p = 1$ calculated above.

The equality $\sum c_i = 1/3$ means that the gravitational potential is unbounded and behaves (up to a factor 4/3) like the Newtonian potential for

$r \approx 0$. The equation $\Phi + \Psi = 2m/r$ enables us to rewrite the metric as

$$ds^2 = (1 - 2\theta)\left[(1 - 2m/r)dt^2 - (1 + 2m/r)(dx^2 + dy^2 + dz^2)\right],$$

which is the conformally transformed linearized Schwarzschild metric with the conformal factor $1 - 2\theta$, where

$$\theta = \frac{m}{r} \sum c_i e^{-r/l_i}$$

can be expressed as functional of the curvature scalar, this is the linearized version of the conformal transformation theorem, cf. SCHMIDT (1990).

For an arbitrary matter configuration the gravitational potential can be obtained by the usual integration procedure.

Appendix

For general p and characteristic lengths l_i fulfilling $0 < l_0 < l_1 < \ldots < l_p$ we write the Lagrangian as

$$\mathcal{L} = R - \frac{R}{6}\Big[(l_0^2 + \ldots + l_p^2)R + (l_0^2 l_1^2 + l_0^2 l_2^2 \ldots + l_{p-1}^2 l_p^2)\Box R + (l_0^2 l_1^2 l_2^2 + \ldots + l_{p-2}^2 l_{p-1}^2 l_p^2)\Box^2 R + \ldots + l_0^2 \cdot l_1^2 \cdot \ldots \cdot l_p^2 \Box^p R\Big]$$

the coefficients in front of $\Box^i R$ in this formula read

$$\sum_{0 \le j_0 < j_1 < \ldots < j_i \le p} \prod_{m=0}^{i} l_{j_m}^2 .$$

Using this form of the Lagrangian, the gravitational potential of a point mass reads

$$\Phi = \frac{m}{r}\left[1 + \frac{1}{3}\sum_{i=0}^{p}(-1)^{i+1}\prod_{j \neq i}\left|\frac{l_j^2}{l_i^2} - 1\right|^{-1} e^{-r/l_i}\right],$$

$$\Psi = \frac{m}{r}\left[1 - \frac{1}{3}\sum_{i=0}^{p}(-1)^{i+1}\prod_{j \neq i}\left|\frac{l_j^2}{l_i^2} - 1\right|^{-1} e^{-r/l_i}\right].$$

For a homogeneous sphere of radius r_0 and mass m we get

$$\Phi = \frac{m}{r}\left[1 + \frac{1}{r_0^3}\sum_{i=0}^{p} e^{-r/l_i}\, l_i^2\, \tilde{c}_i \left(r_0 \cosh(r_0/l_i) - l_i \sinh(r_0/l_i)\right)\right] ,$$

where

$$\tilde{c}_i = (-1)^{i+1} \prod_{j \neq i} |\frac{l_j^2}{l_i^2} - 1|^{-1} .$$

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Remark added in proof: The cosmological consequences of these sixth-order field equations are recently discussed in A. Berkin and K. Maeda (Phys. Lett. B **245**, 348; 1990) and S. Gottlöber, V. Müller and H.-J. Schmidt (1990; preprint PRE-ZIAP 90-16).

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